Tutorial 7

① Go on studying the geodesics on a circular cone:

$$X(u,v) = (u\cos v,\ u\sin v,\ u), \quad u>0$$

$$(g_{ij}) = \begin{pmatrix} 2 & 0 \\ 0 & u^2 \end{pmatrix}.$$

$$\begin{cases} u'' - \frac{u}{2}v'^2 = 0 \\ v'' + \frac{2}{u}u'v' = 0 \end{cases} \quad (*)$$

$(*) \Rightarrow \quad x \triangleq v', \quad x' + 2(\log u)'x = 0.$

$$(e^{2\log u}x)' = e^{2\log u}\left(2\frac{u'}{u}x + x'\right) = 0$$

$$\Rightarrow \quad e^{2\log u}x \equiv C$$

$$\Rightarrow \quad v' = x = \frac{c}{u^2} \quad \text{assume } c \neq 0$$

Unit speed $\Rightarrow \quad 1 = 2u'^2 + u^2v'^2 = 2u'^2 + \frac{c^2}{u^2}$

$$\Rightarrow \quad u' = \pm\frac{1}{\sqrt{2}}\sqrt{1-\frac{c^2}{u^2}}, \quad u \geq |c| > 0, \quad u>0.$$

$$\Rightarrow \quad \frac{dv}{du} = \frac{\sqrt{2}\,c}{\sqrt{u^2(u^2-c^2)}} = \frac{\sqrt{2}\,c}{u\sqrt{u^2-c^2}}, \quad u>|c|$$

$\frac{1}{\sqrt{2}}\sqrt{1-\frac{c^2}{u^2}}, \quad u'' = \frac{u}{2}v'^2 \geq 0$

$$V = \sqrt{2}\,c\int \frac{1}{u\sqrt{u^2-c^2}}\,du$$

$$= \sqrt{2}\,c\left(\frac{1}{c}\sec^{-1}\frac{u}{|c|} + C_0\right)$$

$$= \sqrt{2}\sec^{-1}\frac{u}{|c|} + C_1$$

$\Rightarrow$

$$u(s) = |c|\sec\left(\tfrac{1}{\sqrt{2}}V(s) + C_2\right).$$

If $c=0$ $\Rightarrow$ ($V \equiv$ Constant)

Fact: u-parameter curves are geodesics on the cone. ($\star$)

Or:

$\alpha(u) = X(u, v_0)$, $v_0 =$ fix.

$$\alpha(s) = \left(\tfrac{1}{\sqrt{2}}s\cos v_0,\ \tfrac{1}{\sqrt{2}}s\sin v_0,\ \tfrac{1}{\sqrt{2}}s\right), \quad s = \text{arc-length}.$$

$$\alpha' = \tfrac{1}{\sqrt{2}}(\cos v_0,\ \sin v_0,\ 1)$$

$$\alpha'' = (0,0,0) \quad \Rightarrow (\star).$$

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② Polar coordinate (r,θ) for the punctured flat plane

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} dx = \cos\theta\,dr - r\sin\theta\,d\theta \\ dy = \sin\theta\,dr + r\cos\theta\,d\theta \end{cases}$$

$$g = dx^2 + dy^2 = dr^2 + r^2 d\theta^2 \quad \text{i.e. } (g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

So the origin is a singularity.

$$\Gamma^1_{11} = \frac{1}{2} g^{1i}\left(2g_{1i,1} - g_{11,i}\right)$$

$$= \frac{1}{2} g^{11} g_{11,1} = \frac{1}{2}\cdot 0 = 0$$

$$\Gamma^2_{11} = \frac{1}{2} g^{22}\left(2g_{12,1} - g_{11,2}\right) = 0$$

$$\Gamma^1_{21} = \Gamma^1_{12} = \frac{1}{2} g^{11}\left(g_{11,2} + g_{12,1} - g_{12,1}\right) = 0$$

$$\Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{2} g^{22}\left(g_{12,2} + g_{22,1} - g_{12,2}\right) = \frac{1}{2} r^{-2}\cdot 2r = \frac{1}{r}$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}\left(2g_{12,2} - g_{22,1}\right) = \frac{-1}{2}\cdot 2\cdot r = -r$$

$$\Gamma^2_{22} = \frac{1}{2} g^{22}\left(g_{22,2}\right) = \frac{1}{2}\cdot 0 = 0.$$

Recall geodesic equation: $\dfrac{d^2u^k}{ds^2} + \Gamma^k_{ij}\dfrac{du^i}{ds}\dfrac{du^j}{ds} = 0$

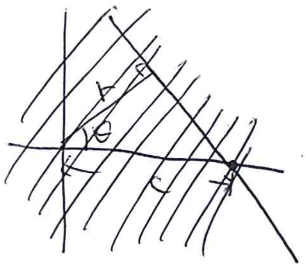
$$\begin{cases} \dfrac{d^2u^1}{ds^2} - r\left(\dfrac{du^2}{ds}\right)^2 = 0 \\ \dfrac{d^2u^2}{ds^2} + \dfrac{2}{r}\dfrac{du^1}{ds}\cdot\dfrac{du^2}{ds} = 0 \end{cases}, \quad \begin{cases} u^1 = r \\ u^2 = \theta \end{cases}$$

i.e

$$\begin{cases} r'' - r(\theta')^2 = 0 \\ \theta'' + \frac{2}{r} r'\cdot\theta' = 0 \end{cases}$$

$\Rightarrow$ $r(s) = |c| \sec(\theta(s) + c_1)$, if $c \neq 0$. $r > |c|$

For $c = 0$, $r(s) = \pm s + c_2$, $\theta = \text{Constant}$.



$\Leftrightarrow$ line equation $y = ax + b$

i.e

$$r\sin\theta = ar\cos\theta + b$$

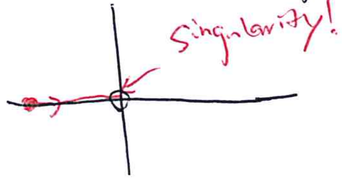
$$r(\sin\theta - a\cos\theta) = b$$

$$r\left(\frac{1}{\sqrt{1+a^2}}\sin\theta - \frac{a}{\sqrt{1+a^2}}\cos\theta\right) = \frac{b}{\sqrt{1+a^2}}, \quad \cos\beta = \frac{1}{\sqrt{1+a^2}}$$

$$r(\sin(\theta - \beta)) = \frac{b}{\sqrt{1+a^2}}$$

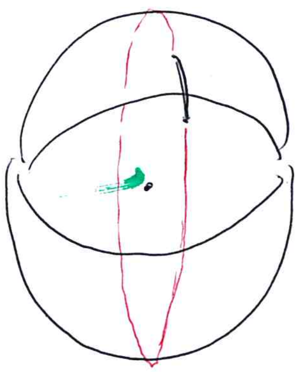
$$r = \frac{b}{\sqrt{1+a^2}} \sec(\theta - \alpha), \quad \alpha = \beta + \frac{\pi}{2}.$$

Rmk : In this example, we see that a geodesic may not be def. on $[0, \infty)$. Consider



③ Locally, geodesic is the shortest curve joining 2 pts.

But this is not true globally.



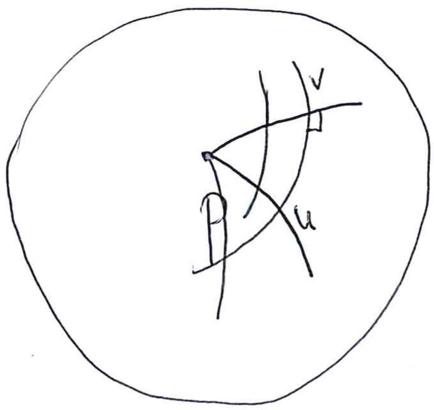
Locally, one can find a coordinate patch of $p \in M$ = a surface

s.t

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & G \end{pmatrix} \qquad \text{i.e.} \quad |X_u|^2 = 1, \quad \langle X_u, X_v \rangle = 0$$

$$|X_v|^2 = G$$

We call it geodesic polar coordinate of p.



$\alpha(u) = X(u, v_0)$ = geodesic

Pf: $\alpha' = X_1, \ \alpha'' = X_{11}$

$\Rightarrow |\alpha'|^2 = |X_1|^2 = 1 \Rightarrow$ param. by arc-leng

$\langle \alpha'', X_1 \rangle = \langle X_{11}, X_1 \rangle = \frac{1}{2} \langle X_1, X_1 \rangle_1 = 0$

$\langle \alpha'', X_2 \rangle = \langle X_{11}, X_2 \rangle = \langle X_1, X_2 \rangle_1 - \langle X_2, X_{21} \rangle$

$= -\frac{1}{2} \langle X_1, X_1 \rangle_2 = 0$

$\Rightarrow \quad (\alpha'')^T = 0 \Rightarrow \alpha$ is a geodesic.

Now we consider another curve $\beta(s)$, $s \in [0, L]$, where L is the length of $\alpha(u)$ s.t.

$$\begin{cases} \beta(0) = \alpha(0) = X(0, V_0) \\ \beta(L) = \alpha(L) = X(L, V_0) \end{cases}$$

Write $\beta(s) = X(u(s), v(s))$, So $u(0) = 0$, $u(L) = L$.

$$\beta' = X_1 \cdot u' + X_2 \cdot v'$$

$$\Rightarrow \quad L(\beta) = \int_0^L \sqrt{\langle \beta', \beta' \rangle}\, ds$$

$$= \int_0^L \sqrt{u'^2 + G v'^2}\, ds \geq \int_0^L |u'| ds \geq \int_0^L u' ds$$

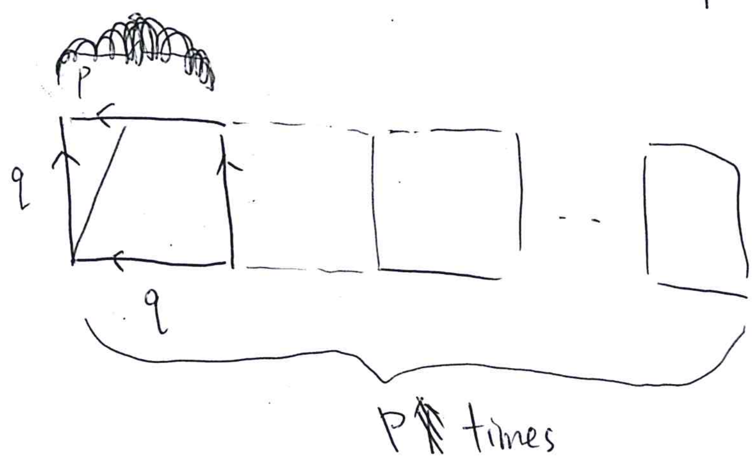
$$= u(L) - u(0) = L.$$

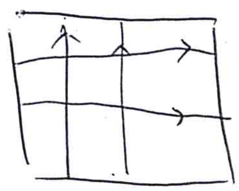
$$L(\beta) = L(\alpha) \iff u' > 0, \quad v' \equiv 0$$

$$\Rightarrow \beta(s) = X(u(s), V_0)$$

i.e. $\beta = \alpha$ up to reparametrization.

(4) Geodesics on abstract flat square torus :





There are closed geodesics which have rational slopes.